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Renormalization of Aubry–Mather Cantor Sets

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Let f be a two-dimensional area-preserving twist map. Given an irrational rotation number ω in the rotation interval of f, there is an invariant recurrent set on which f preserves the circular ordering and which has rotation number ω. For large nonlinearity, the parameter regime we are interested in, this set is a Cantor set. We show that well-ordered (minimizing) sets with rotation numbers close to ω are exponentially close to the Cantor set under study. The detailed configuration of well-ordered (minimizing) sets is universal and depends on one parameter, namely the Lyapunov coefficient of the Cantor set. There is a quantitative correspondence between this and similar behavior in the noninvertible circle map.

KEY WORDS: Twist maps; Aubry–Mather Cantor sets; incommensurate minimizing states without translational invariance; first return maps; scaling; exponential approximation by periodic orbits.

1. INTRODUCTION

As a simple model of irregular behavior, area-preserving twist maps of the cylinder present us with considerable mathematical problems. On the other hand, some of their aspects are reasonably well understood. In particular, order-preserving behavior is at least qualitatively well understood in terms of KAM theory(1) and in terms of minima of a certain functional (Cantori or Aubry–Mather sets(2,5,6,13,21)). The purpose of this work is to obtain a precise, quantitative insight into the way these order-preserving orbits with rational rotation number accumulate on the Aubry–Mather sets that have irrational rotation number *once these are Cantor sets*. We will conclude that order-preserving behavior in twist maps is quantitatively the same as similar behavior in noninvertible maps of the circle.

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In the presence of small nonlinearity in a two-dimensional area-preserving twist map f, the behavior of the orbits is to some extent organized by the order-preserving orbits. In this case, the KAM theorem implies that an invariant (KAM) curve can be assigned to most irrational rotation numbers. These are continuous and Lipschitz and on them, f preserves the circular ordering (order-preserving orbits). The invariant regions between these curves (Birkhoff zones) typically consist of hyperbolic order-preserving periodic orbits alternating with elliptic islands (called resonances), as was already observed by Poincaré.

Our notation will be as follows. The minimizing order-preserving orbits which are recurrent will be denoted by E_ω, where ω is the rotation number associated with the set. Homoclinic minimizing orbits are denoted by $E_{p/q-}$ or $E_{p/q+}$ (we will not consider irrational homoclinic orbits).

Hausdorff limits (H-lim) of these sets are well defined,[13]

$$\begin{aligned} \underset{\alpha \downarrow p/q+}{H\text{-lim}}\, E_\alpha &= E_{p/q} \cup E_{p/q+} = \operatorname{clos}(E_{p/q+}) \\ \underset{\alpha \uparrow p/q-}{H\text{-lim}}\, E_\alpha &= E_{p/q} \cup E_{p/q-} = \operatorname{clos}(E_{p/q-}) \end{aligned} \tag{1.1a}$$

For ω irrational

$$E_\omega \subseteq \underset{\alpha \to \omega}{H\text{-lim}}\, E_\alpha \tag{1.1b}$$

When the nonlinearity is increased, the invariant curves break up into Cantor sets with irrational rotation number. The magnitude of the nonlinearity at which a curve breaks depends on the number-theoretic properties of its rotation number.[11,13] Once the curve has broken up, it appears to have a positive Lyapunov exponent.[9] To our knowledge, the only case for which hyperbolicity of E_ρ has been proven is for the standard map with large nonlinearity.[4]

Assumption. ("Hyperbolicity") Note that, if E_ρ is a Cantor set, then the union of the Aubry–Mather sets sufficiently close to E_ρ is a hyperbolic set (which we denote by H, see refs. 8 and 21). This implies that each point x of H has stable and unstable manifolds $W^s(x)$ and $W^u(x)$, which depend continuously on x, and satisfy:

$$\text{if} \quad y \in W^s(x): \qquad d(f^n(x), f^n(y)) < C\lambda^{-n}(x)\, d(x, y) \qquad \text{as} \quad n \to \infty$$

$$\text{if} \quad y \in W^u(x): \qquad d(f^{-n}(x), f^{-n}(y)) < C\lambda^{-n}(x)\, d(x, y) \qquad \text{as} \quad n \to \infty$$

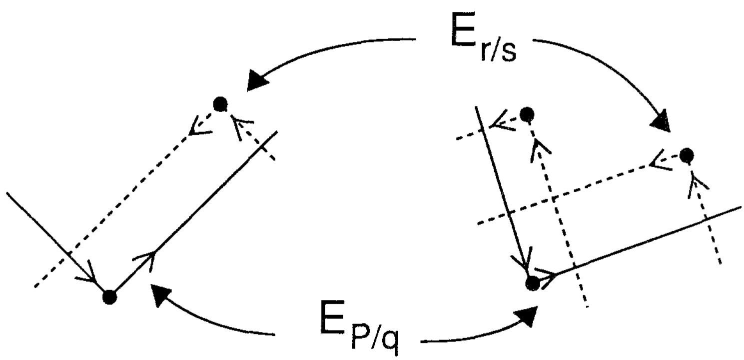

Fig. 1. The diamond conjecture.

where C is a constant and the "eigenvalues" $\lambda(x)$ are uniformly greater than one. (For the definition of uniform hyperbolicity see ref. 7.)

The process of breaking up is important for two reasons. First, it implies the loss of global stability, as eventually no invariant curves confine the behavior of orbits. The diffusion across the Cantor sets is described in refs. 12 and 21. Second, the breaking up implies the loss of translation invariance of certain minimizing orbits.[2] MacKay[10] summarizes a renormalization group approach to finding the geometric properties of the map at the point where invariant curves break up. In this work, we develop a renormalization approach that encodes the geometrical properties of the E_ρ after they have broken up, thereby explaining the "bunching" observed by Chen *et al.*[3]

Our approach is based on the following result.[22]

Diamond Configuration. If E_α is hyperbolic, then for r/s and p/q ($r/s < \alpha < p/q$) close enough to α, E_α is contained in a region J that is the union of "diamonds" and whose boundary is formed by local stable and unstable manifolds to $E_{p/q}$ and $E_{r/s}$ only. (See Fig. 1.)

In the next section we give a brief outline of some of the topological properties of irrational Aubry–Mather sets. In Section 3, we discuss how to set up a "first return map" in a neighborhood of the Aubry–Mather set. This is done in close analogy with renormalization practices in one dimension. Some properties of this map will be briefly discussed in Section 4. Finally, in Section 5, we perform the calculations necessary to see how the rational orbits approach the Cantor set.

2. THE TOPOLOGY OF AUBRY–MATHER SETS

We summarize previous work on Aubry–Mather sets[21,22] that prepares the stage for the ideas to be developed in this paper. For proofs and further results, we refer the reader to refs. 21 and 22.

First, there is a natural ordering of the sets E_ρ. If $\rho > \beta$, then E_ρ lies above E_β. Here "above" has the following meaning. Take a rational p/q so that $\rho > p/q > \beta$. Denote the minimizing q periodic orbit associated with that rational by $E_{p/q}$ and the minimizing homoclinic orbit by $E_{p/q+}$ and $E_{p/q-}$. We construct a curve $\gamma(p/q+)$ as follows. Choose a point s of $E_{p/q+}$ and connect s along the invariant manifolds to the neighboring points of $E_{p/q}$. Repeat this q times so that γ separates the cylinder. If γ is a non-self-intersecting curve, then E_ρ lies above it and E_β below. In case γ is self-intersecting, the meaning of "above" is somewhat more subtle.

For f generic, there is a set I of rotation numbers which is open and dense in the rotation interval J, such that if ρ is in I, then E_ρ is hyperbolic (this had already been proven by Le Calvez[8]). In some cases I equals J.[4] We can define a hyperbolic set $H = \bigcup_{\rho \in K} E_\rho$, where $K \subseteq I$ is a compact subset of I. The hyperbolicity together with the previous result leads to the "diamond" configuration discussed in the introduction. Hyperbolicity will be assumed from now on without being mentioned explicitly.

For the study of transport across the Aubry–Mather sets, it is important to know how many gap orbits a minimizing Cantor set has. Under certain conditions for f, there is only one gap orbit.

Let G_i be a gap in E_{p_i/q_i+} chosen in such a way that $\lim G_i = G$, where G is a gap in E_α and $\alpha = \lim p_i/q_i$ is irrational. Denote the pieces of invariant manifolds connecting G_i, resp. G, by W_i^{u} and W_i^{s}, resp. W^{u} and W^{s}. The fact that the Cantor sets have only one gap can also be used to establish that $\lim W_i^{\mathrm{u}} = W^{\mathrm{u}}$ and $\lim W_i^{\mathrm{s}} = W^{\mathrm{s}}$. This implies that the region that diffuses across an Aubry–Mather Cantor set is a limit of resonance overlaps.

From now on, we let $\lambda(\rho)$ denote the Lyapunov coefficient >1 for an order-preserving minimizing set E_ρ. Since these sets are uniquely ergodic with invariant probability measure $\mu(\rho)$,[13] $\lambda(\rho)$ is well defined and constant μ almost everywhere. Let E_ω be a hyperbolic minimizing set with irrational rotation number; then $\lambda(\rho)$ is continuous at $\rho = \omega$.

Finally, we mention a lemma that will be needed later. Let $x \in E_\rho$ and let v be the vertical directed from x upward. Going around *clockwise*, beginning at v, the first invariant direction is unstable. We will denote this direction by $W_1(x)$ and the subsequent ones by $W_i(x)$, where $i \in \{2, 3, 4\}$.

3. THE FIRST RETURN MAP

In this section we will define the first return map from which we will later calculate how periodic orbits approximate the Cantor set.

Three remarks are in order before we do the construction. Our construction is entirely analogous to the construction of the first return map of a one-dimensional piecewise linear noninvertible circle map. We will therefore see, and this is the second remark, that our results are essentially the same as those for this class of circle maps.[18,19] Here we need some assumptions to control the nonlinearities. We have chosen them in such a way that their natural one-dimensional analogues can be proven to hold.[16] Third, our construction makes use of invariant manifolds to the Cantor set. Therefore we cannot apply it directly to the critical case of the breaking up of invariant circles.[10]

We will assume from now on that f has a single minimizing p/q orbit $E_{p/q}$ for all rotation numbers in the rotation interval of the map. This is true for generic f.

Let us consider one of the diamonds K_i of Fig. 1. Such a diamond is formed by two minimizing hyperbolic periodic points s_i and s_{i+1} of rotation number p_i/q_i and p_{i+1}/q_{i+1} and their local stable and unstable manifolds. Notice that the stability type of the manifolds is determined by the lemma mentioned in Section 2. From now on, we will take ω to be a fixed irrational rotation number with continued fraction expansion $\omega = [\alpha_1, \alpha_2,...]$ and p_i/q_i its continued-fraction approximants.

We define a first return map on (a subset of) K_i to K_i as follows (see Fig. 2). Let

$$A_i = f^{-q_i}(K_i) \wedge K_i$$

$$B_i = f^{-q_{i+1}}(K_i) \wedge K_i$$

Define R_i: $A_i \cup B_i \to K_i$ as f^{q_i} on A_i and $f^{q_{i+1}}$ on B_i and assume that A_i and B_i are single strips. It is important to observe that this definition is only correct if A_i does not overlap B_i. This is not much of a restriction, since, for i large enough, f^{q_i} has large eigenvalues. On the other hand, bearing in mind that we are only interested in those orbits of R_i whose projection preserves the circular ordering, we could have restricted to a much smaller set, thereby avoiding the problem altogether.

Since the E_{p_i/q_i} uniformly approximate E_ω [see Eq. (1.1)], the diameter of the diamonds K_i goes to zero as $i \to \infty$. It then follows by standard hyperbolic theory[7] that the invariant manifolds which form the boundary of K_i uniformly approximate straight line segments.

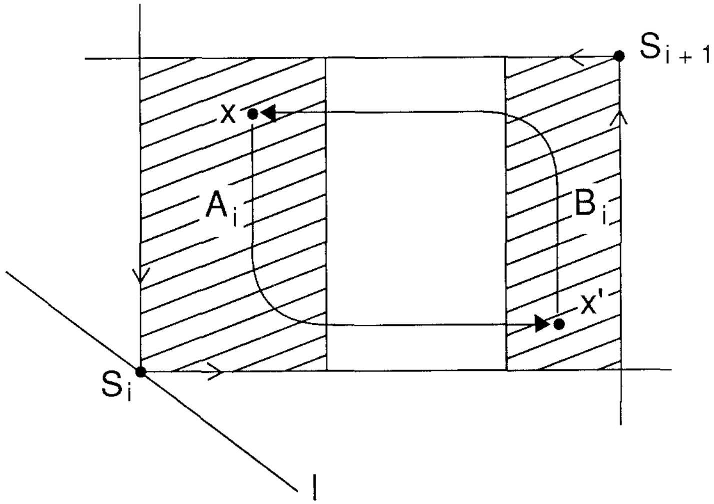

Fig. 2. The action of the R. The unshaded region is mapped outside K.

We know by the Hartman–Grobmann theorem that f^{q_i} and $f^{q_{i+1}}$ can be linearized in s_i (respectively s_{i+1}). In order to control the nonlinearities of the return map, we introduce the following somewhat stronger assumption:

Assumption 3.1. $f^{-q_i} \cdot Df^{q_i}(s_i)$ restricted to the nonwandering set in A_i and $f^{-q_{i+1}} \cdot Df^{q_{i+1}}(s_{i+1})$ restricted to the nonwandering set in B_i converge to the identity in a Hölder norm.[16]

As stated before, $\lambda(\rho)$ is continuous at $\rho = \omega$. Again, we conjecture a stronger result.

Assumption 3.2. Let ω be any irrational and p_i/q_i its continued-fraction approximants. There is a constant $C(\omega)$ not dependent on q_i such that

$$|\lambda(\omega) - \lambda(p_i/q_i)| < C(\omega)/q_i$$

We note that in the case of the standard map with sufficiently large perturbation these assumptions have been proved in ref. 16.

Suppose, then, that p_i/q_i is less than ω and that of all the order-preserving orbits intersecting K_i, we want to find the one that is periodic with rotation number $(p_i + \alpha_{i+2} p_{i+1})/(q_i + \alpha_{i+2} q_{i+1}) = p_{i+2}/q_{i+2}$ (the next continued-fraction approximant). Then there is a q_{i+2}-periodic point x in K_i which does not leave K_i under repeated application of R_i and satisfies (see Fig. 2) $x \in A_i$ and $x' \in B_i$ with $f^{q_i}(x) = x'$ and $f^{\alpha_i - 2q_i + 1}(x') = x$. It follows that x is an $(\alpha_{i+2} + 1)$-periodic orbit of R_i. The assumptions imply (see Section 5) that we can replace these iterates by their respective linearizations at s_i and s_{i+1}.

To study this map conveniently, one can linearly change coordinates in K_i to $(x, y) \in [0, 1] \times [0, 1]$ in such a way that s_i corresponds to $(0, 0)$ and s_{i+1} to $(1, 1)$. Df^{q_i} contracts vertically and expands horizontally at a rate $\tau_i = \lambda(p_i/q_i)^{q_i}$ and similarly for $Df^{q_{i+1}}$ at a rate τ_{i+1}. Thus, R_i can be written as

$$f_i: \quad \begin{cases} x' = \tau_i x \\ y' = y/\tau_i \end{cases} \qquad \text{for} \quad x \geqslant 1 - 1/\tau_{i+1} \tag{3.1b}$$

$$g_i: \quad \begin{cases} x' = 1 - \tau_{i+1}(1 - x) \\ y' = 1 - (1 - y)/\tau_{i+1} \end{cases} \qquad \text{for} \quad x \geqslant 1 - 1/\tau_{i+1} \tag{3.1b}$$

4. PROPERTIES OF THE FIRST RETURN MAP

We indicate some of the properties of the first return map R defined in the previous section. The results in this section were previously discussed in refs. 17 and 18, where they arose in connection with the study of non-invertible circle maps.

Clearly, the map R_i is one out of a two-parameter family of maps, the parameters being τ_i and τ_{i+1}. In Section 5, we will show that, for our purposes, it suffices to only consider a one-parameter family, namely

$$R(\tau): \quad \begin{cases} x' = \tau x \bmod(\tau - 1) \\ y' = y/\tau + (\tau - 1)/\tau \cdot \text{int}\{x(\tau - 1)/\tau\} \end{cases} \tag{4.1}$$

(Here, we have assumed that $\tau > 2$.) $R(\tau)$ is the linear map with eigenvalues τ and $1/\tau$ and fixing the points $(0, 0)$ and $(1, 1)$.

We note that $R(\tau)$ restricted to its recurrent set is conjugate to the 2 shift on $\{0, 1\}^{Z}$, the space of bi-infinite binary sequences

$$(\cdots i_{-1} i_0 \cdot i_1 i_2 \cdots)$$

The conjugacy is given by the following formulas:

$$x = C_1(\tau) \sum_{0}^{-\infty} i_j/\tau^{|j|}$$

$$y = C_2(\tau) \sum_{0}^{-\infty} i_j/\tau^{|j|}$$

Note that the rotation number in this context corresponds to $\lim_n (1/n) \sum_1^n i_j$. The notion of circular ordering can be defined by projecting orbits to a line l through the point $(0, 0)$ that does not intersect the unit square, as drawn in Fig. 2.

The following results can be found in the articles quoted in the introduction to this section. They are proven there in the context of linear maps, but carry over to general hyperbolic return maps that admit the same symbolic dynamics.

Result 1. There is a *constructive* procedure to find the set A_l of all recurrent points whose orbits under R projected to l are order preserving. Moreover, the sets A_{l_1} and A_{l_2} are equal for each l as defined above. We denote this set simply by A.

Result 2. For each rotation number p/q in $[0, 1]$, A contains a single order-preserving orbit $E_{p/q}$.

Result 3. The Hausdorff limit set of the sequence E_{p_i/q_i} is equal to E_ω for $p_i/q_i \to \omega$.

Result 4. Consider l as the new x axis. There exists a family of continuous, non-self-intersecting Lipschitz (as graphs on the new x axis) curves $\gamma(\rho)$ such that for each ρ, $E_\rho \subset \gamma(\rho)$ and $\gamma(\rho_1) \wedge \gamma(\rho_2) = \varnothing$ iff $\rho_1 \neq \rho_2$. Moreover, $\gamma(\rho_1)$ lies above $\gamma(\rho_2)$ iff $\rho_1 > \rho_2$.

Figure 3 is a picture of the set A for $R(2)$, and Fig. 4 depicts E_ω for

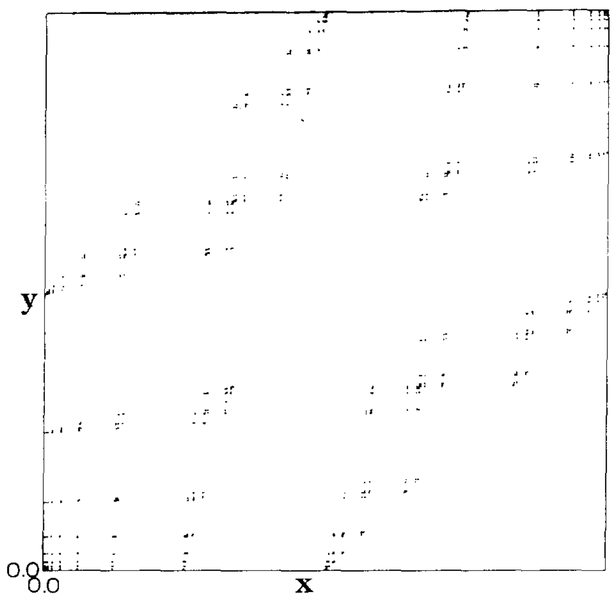

Fig. 3. The order-preserving orbits of $R(2)$.

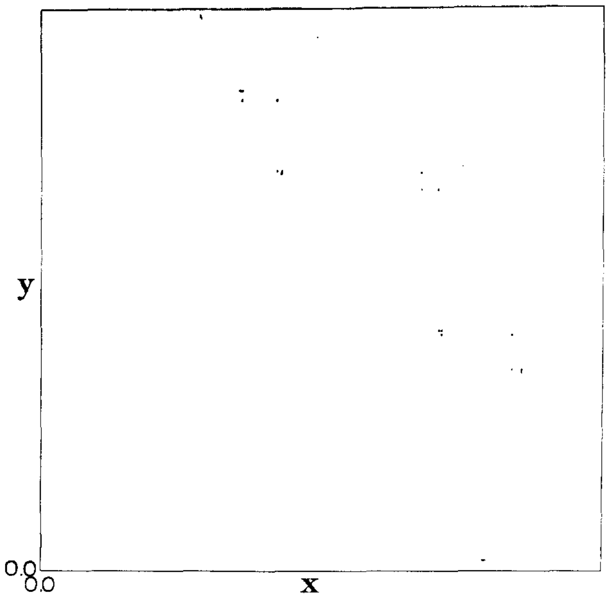

Fig. 4. The golden mean recurrent set of $R(2)$.

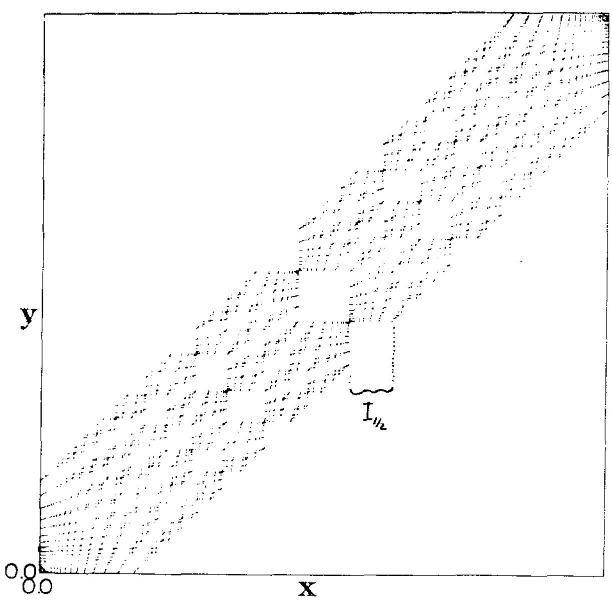

Fig. 5. The order-preserving orbits of $R(1.2)$. The indicated distance is equal to $I_{1/2}$.

that map, where ω is the golden mean. Figure 5 pictures the set A again, but this time for the map $R(1.2)$.

A least remark concerns the methods used to prove the above results. It is interesting to know that these make no reference to general results about twist maps, such as the theorems of Mather and Aubry.

5. APPROXIMATION OF CANTOR SETS

In this section, we study analytically how the E_{p_i/q_i} approach a Cantor set E_ω as $p_i/q_i \to \omega$. To simplify the exposition, we will take ω to be the golden mean $\omega = [1, 1,...]$. In the treatment we give, the convergence of E_{p_i/q_i} close to a Cantor set E will be completely determined by a single parameter, namely, $\lambda(\omega)$, the Lyapunov coefficient associated with E_ω. In this sense, we can speak of universality. It will also turn out that this behavior is quantitatively the same as the corresponding behavior in non-invertible circle maps. As a result of these considerations, it will become clear why and how high periodic orbits tend to group together as the parameter mentioned above increases.

We are interested in characterizing certain properties of the recurrent orbits of the map R_i defined on K_i. As discussed in the previous section, we can assign rotation numbers to order-preserving orbits of R_i. These numbers bear a simple relation to the rotation number of the same orbit under f. Recall that the periodic orbits in the boundary points of K_i have rotation number p_i/q_i, resp. p_{i+1}/q_{i+1}, where we assume that the latter is the larger of the two. From the definition of R_i, one concludes easily that a rotation number $\alpha = r/s$ (in lowest terms) of R_i satisfies the following correspondence:

$$\frac{r}{s}\ \{\text{of } R_i\} \leftrightarrow \frac{(s-r)\,p_i + rp_{i+1}}{(s-r)\,q_i + rq_{i+1}}\ \{\text{of } f\}$$

$$= \frac{(1-\alpha)\,p_i + \alpha p_{i+1}}{(1-\alpha)\,q_i + \alpha q_{i+1}}\ \{\text{of } f\} \tag{5.1}$$

We now proceed to study the scalings associated with the Cantor set E_ω. The first step entails a substantial simplification of the problem. We reduce the study of the sequence of maps R_i to the study of one single map $R(\tau)$ (with one parameter) as follows.

The eigenvalues of R_i at s_i is $\lambda(p_i/q_i)^{q_i}$ and at s_{i+1} it is $\lambda(p_{i+1}/q_{i+1})^{q_{i+1}}$. The second assumption in Section 3 now implies that

$$\left[1 - \frac{C(\omega)}{\lambda(\omega)\,q_i}\right]^{q_i} < \left[\frac{\lambda(p_i/q_i)}{\lambda(\omega)}\right]^{q_i} < \left[1 + \frac{C(\omega)}{\lambda(\omega)\,q_i}\right]^{q_i} \tag{5.2}$$

Therefore, $[\lambda(p_i/q_i)/\lambda(\omega)]^{q_i}$ is contained in an interval $(1/k(\omega), k(\omega))$ not dependent on q_i.

Now consider $R(\lambda(\omega))$ on the unit square, as in (4.1) with $\tau = \lambda(\omega)$. One observes that, possibly after a linear change of coordinates, the first return map of $R(\lambda(\omega))$ restricted to the square K_i is precisely R_i plus a bounded error. In other words, $R(\lambda(\omega))$ is a linear map that mimics, quantitatively and qualitatively, the order-preserving behavior of f restricted to some diamond K_i, but *not* necessarily outside that diamond.

Now we are in a position to calculate scalings. Consider the following quantity:

$$I_{p/q} = \max_{E_{p/q+}} \min_{E_{p/q}} \{ |\pi(p_1) - \pi(p_2)| \mid p_1 \in E_{p/q+}, p_2 \in E_{p/q} \} \tag{5.3}$$

where π is the projection along the stable direction. $I_{p/q}$ is the largest distance between homoclinic points and their nearest periodic point in $E_{p/q+}$ (see Fig. 5). One concludes from ref. 18 that $I_{p/q}$ satisfies

$$|I_{p/q}| = C_{p/q} \cdot 1/(\lambda^q - 1)$$

where $C_{p/q}$ is uniformly bounded away from zero and infinity. Set

$$\beta_i - \beta_{i+1} = \frac{1}{2}\left(|I_{p_i/q_i}| + |I_{p_{i+1}/q_{i+1}}|\right) + \sum' |I_{r/s}|$$

where the summation runs over all rationals r/s, $p_i/q_i < r/s < p_{i+1}/q_{i+1}$. Now, define the "exponent" δ, following Shenker,[15]

$$\delta_i = \left| \frac{\beta_{i-1} - \beta_i}{\beta_i - \beta_{i+1}} \right| \tag{5.4}$$

Then δ_i grows as $K_{p_i/q_i} \cdot \lambda^{+q_i - q_{i-1}}$, where the K are uniformly bounded (see ref. 18, where this calculation is done for circle maps). We note here that for i large enough, $|\beta_i - \beta_{i+1}|$ is an upper bound for the distance of s_i to E_ω.

An alternative way to calculate how distances in K_i scale is achieved by using formula (3.1). To approximate the Cantor set by its continued-fraction approximants, we look for a 2-periodic orbit in each K_i. That is, in formula (3.1) we have

$$f_i \circ g_i(x, y) = (x, y)$$

This is easily solved. Because f_i and g_i are approximately linear (modulo a bounded error in the derivative) and both f_i and g_i have to map the point in narrow strips, namely B_i and A_i, the orbit is (see Fig. 6)

$$p_1 = \left(\frac{1}{\tau_i + k_i(\omega)}, 1 - \frac{1}{\tau_{i+1} + k_i(\omega)}\right)$$

$$p_2 = \left(1 - \frac{1}{\tau_{i+1} + k_i(\omega)}, \frac{1}{\tau_i + k_i(\omega)}\right)$$

where $k_i(\omega)$ is a uniformly bounded error. To do the next step of the calculation, we set s_{i+2} equal to p_2. Define

$$\delta'_{i+1} = \left|\frac{\pi(s_i) - \pi(s_{i+1})}{\pi(s_{i+1}) - \pi(s_{i+2})}\right|$$

Then by straightforward calculation, one arrives at

$$\delta'_i \sim \gamma_{p_i/q_i} \cdot \lambda(\omega)^{+q_i} \qquad (\text{as } i \to \infty)$$

and γ uniformly bounded. This is the length scaling of the *shortest* edges of the diamonds. (Note that these scalings are not quite the same ones as before.) By the diamond conjecture, which says that the Cantor set is contained in the diamonds, it also equals the scaling of the areas of successive diamonds K_{i-1} and K_i.

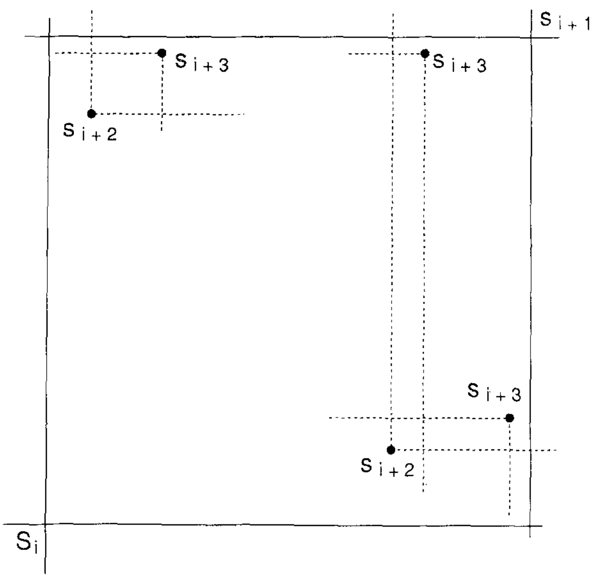

Fig. 6. The nested renormalization construction.

We now proceed to find the asymptotic behavior of the Hausdorff distance $d(E_{p_i/q_i}, E_\omega)$. Consider the domain of R_i and set up the symbolic dynamics as described in Section 4: let x be a point in the recurrent set of R_i; then

$$i_n = 0 \qquad \text{if} \quad x \in A_i$$

$$i_n = 1 \qquad \text{if} \quad x \in B_i$$

Since we discuss the golden mean in this section, we can substitute q_{n-1}/q_n for r/s and q_{i-1} for p_i in (5.1). We can then translate the rotation numbers under R_i and under f as follows:

$$\frac{q_{n-1}}{q_n}\{\text{of } R_i\} \leftrightarrow \frac{(q_n - q_{n-1})\, q_{i-1} + q_{n-1} q_i}{(q_n - q_{n-1})\, q_i + q_{n-1} q_{i+1}}\{\text{of } f\}$$

$$= \frac{q_{i+n-1}}{q_{i+n}}\{\text{of } f\} \tag{5.5}$$

Consider the q_{n-1}/q_n and ω order-preserving sets of R_i, $E_{q_{n-1}/q_n}\{R_i\}$ and $E_\omega\{R_i\}$. The shortest binary sequence on which two points u and v in $E_{q_{n-1}/q_n}\{R_i\}$ and $E_\omega\{R_i\}$, respectively, can differ has length q_n (see Appendix). Since we are dealing with bi-infinite sequences, this means that there are two such points u and v with different values for either $i_{-\text{int}[q_n/2]}$ or $i_{\text{int}[q_n/2]}$. So

$$f(i) \cdot d(R_i^{q_n/2}(u), R_i^{q_n/2}(v)) = d(A_i, B_i)$$

where $f(i)$ is a factor that corrects for the fact that we replaced int$[q/2]$ by $q/2$. Recall that under the assumptions made in Section 3 and Eq. (5.2), R_i stretches horizontal distances in A_i by a factor $\lambda(\omega)^{q_i}(1+\delta)$ and in B_i by a factor $\lambda(\omega)^{q_{i+1}}(1+\delta)$, where δ is uniformly bounded. Moreover, we can choose i so big that δ is arbitrarily close to zero. From the foregoing [using (5.5)], we then obtain

$$(1+\delta)^{q_n/2}\, \lambda(\omega)^{(q_n - q_{n-1})q_i/2} \lambda(\omega)^{q_{n-1}q_{i+1}/2} f(i)\, d(u, v) = d(A_i, B_i)$$

It is not hard to see that under the assumptions made, the factor $d(A_i, B_i)/f(i)$ is of order at most $\lambda(\omega)^{q_i}$. Since there are finitely many diamonds formed by the invariant manifolds of E_{q_{i-1}/q_i} and $E_{q_i/q_{i+1}}$, one obtains finally that

$$d(E_{q_{i+n-1}/q_{i+n}}, E_\omega) = \lambda(\omega)^{-q_{i+n}/2}(1+\delta)^{q_n/2}$$

where δ may depend on i and n but is uniformly bounded (and small in absolute value).

Finally, we observe that $\lambda(\omega)$ varies as a function of the parameters in the map f. This explains the "bunching" observed numerically by Chen *et al.*[3] Notice that we explain the bunching of continued-fraction approximants and not of Farey approximants, the latter of which are studied by these authors. This difference is essential only when ω is an irrational of unbounded type (that is: the α_i are unbounded). In that case, which is not studied in the quoted paper, the convergence can almost certainly be much worse for the Farey approximants (note that if $\omega < 1/N$, then for all $n \in \{1, 2,..., N\}$ the quotient $1/n$ is a Farey approximant).

The bunching phenomenon can be simulated simply by studying the map $R(\tau)$ for different τ. As an example, we refer to Figs. 3 and 5; in the former figure τ is greater and the orbits cease being distinguishable (with resolution of this picture) at much lower period. This is a direct consequence of the fact that the scalings are infinite, which, in turn, indicates that the hyperbolic sets under consideration have Hausdorff dimension zero (see next section).

6. CONCLUDING REMARKS

We have shown, in general, that minimizing orbits that preserve circular ordering approximate the minimizing Cantor set in an exponential fashion. Moreover, their detailed configuration in the neighborhood of a Cantor set is universal, that is, depends on the single parameter which is the eigenvalue associated with that Cantor set.

To turn the reasoning of this paper into proofs, one has to prove the assumption that $Df^{q_i} \cdot f^{-q_i}$ converges to the identity (as $i \to \infty$), and that the Lyapunov coefficient $\lambda(\rho)$ satisfies the estimate of Assumption 3.2. This is done in a sequel[16] to this work. The only caveat is that the norm referred to in Assumption 3.1 is somewhat more complicated than indicated here.

We point out here that in one dimension the closure of the union of all hyperbolic order-preserving Cantor sets has Hausdorff dimension zero.[20] This, apparently, is also the case for the two-dimensional twist maps discussed in this paper.

We make a final remark about our procedure to calculate the distance scalings in Section 5. We have discussed two different procedures, of which the second appears to be the more natural in our context. However, it is worth pointing out that the formulation of the first procedure does not depend on the presence of a hyperbolic structure. Therefore, it may be that it gives a natural framework for studying critical problems such as the breaking up of invariant curves (for a discussion of that problem see ref. 10).

APPENDIX

Consider the map given by (4.1) and its order-preserving recurrent orbits as described in Section 4. As noted in that section, we can study the symbolic dynamics of these sets. Let $\Sigma(\rho)$ denote the set of bi-infinite binary sequences representing points of orbits with rotation number ρ. If ρ is irrational, we denote by p_n/q_n its continued-fraction convergents. In this Appendix we demonstrate that, for irrational ρ, the shortest finite differing subsequence, that is, one that occurs in *either* $\Sigma(\rho)$ *or* $\Sigma(p_n/q_n)$ (but not in both) has length q_n.

We will use the following result.[17] A sequence is in $\Sigma(\rho)$ if and only if, for all s, *all* of its finite subsequences z of length s satisfy

$$\frac{r}{s} < \rho < \frac{r+1}{s} \Leftrightarrow z \text{ has either } r \text{ or } r+1 \text{ ones}$$

$$\frac{r}{s} = \rho \Leftrightarrow z \text{ has } r \text{ ones}$$

To see that there exists a differing subsequence z^* of length q_n, note that there must be sequences in $\Sigma(\rho)$ (ρ irrational) of length q_n that have $p_n + 1$ ones according to the quoted result. [Otherwise their rotation number (see Section 4) would not be irrational.]

We now have to show that all subsequences of length smaller than q_n that occur in $\Sigma(\rho)$ also occur in $\Sigma(p_n/q_n)$. We argue by contradiction. Let z^* be a differing subsequence in $\Sigma(\rho)$ of length w less than q_n. Then one possibility is that z^* has v ones, and that sequences of length w in $\Sigma(p_n/q_n)$ all have, say, $v-1$ or $v-2$ ones. So by the result

$$\frac{v-2}{w} < \frac{p_n}{q_n} < \frac{v-1}{w} \qquad \text{and} \qquad \frac{v-1}{w} < \rho < \frac{v}{w} \qquad \left(\text{or} \quad \frac{v}{w} < \rho < \frac{v+1}{w}\right)$$

so that $(v-1)/w$ lies between a convergent and the irrational number, which is a contradiction. The other possibility is that z^* has $v-1$ or $v-2$ ones, but that the ordering of the ones is unlike that of any subsequence of length w in $\Sigma(p_n/q_n)$. This, however, implies that the previous possibility holds for some subsequence in z^* of length smaller than w. In this reasoning the roles of ρ and p_n/q_n can be interchanged.

We remark that by a similar method one can see that for every point x in the Cantor set E_ω there is a point y *homoclinic* to E_{q_{n-1}/q_n} whose binary expansion matches that of v on $\{i_{-q_n}, i_{-q_n+1},..., i_{q_n}\}$. The reasoning of Section 5 then yields the result that (with δ uniformly bounded and small)

$$d(x, y) = \lambda^{-q_{i+n}}(1+\delta)^{q_n}$$

that is, the square of the distance to the periodic orbit.

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